Matrix Factorization in Broad Spectrum

A contemporary innovation in the advent of computers is the matrix factorization. It has been making headway for the past fifty years. Its inception was necessitated by a need to accomplish a specific work, progressing from a mere perception of matrices through the gateway of understanding the principles of modern matrices in light of matrix factorization.

A few of the matrix factorizations discussed in the textbook are LU Factorization, QR Factorization, EV Factorization, Orthogonal Factorization, and Singular Value Factorization (SVD). These types of factorizations are presented in view of the tasks they achieve or the role they play in solving problems. For instance, the SVD and EV Factorizations may have factorization as their end goal but this job contributes to an easier way to solve and understand solutions. In this regard, matrix factorizations have opened the doors toward understanding matrix mappings (functions) and the solutions of linear systems.

LU Factorization

Provided a square matrix A or a matrix A with more unknowns than constraints, there exists a way to eliminate the unknowns from a list of equations. This factorization has the form **PA = LU**. To fulfill its conditions for the given matrix A, there must be:

- A permutation matrix **P**
- An invertible lower triangular matrix **L** with ones on the main diagonal. This is the product of the inverses of the Gaussian matrices G_n resulting from multiplying matrix A with each Gaussian matrix G_n because $L = G^{-1}{}_1 * G^{-1}{}_2 * \ldots * G^{-1}{}_n$.
- The upper triangular row echelon form of matrix A, which is **U**.

With the pre-requisites satisfied, LU factorization can solve the linear system Ax = b, taking into account, the form of matrix A above. Thus, Ax = b can have the structure LUx = b. Through forward substitution or back substitution, the computation is implemented.

QR Factorization

Bases are defined by span and linear independence. One of the significant bases is the orthonormal basis $\{q_1,q_2,\ldots,q_n\}$ that can be created by the Gramm-Schmidt process. In this regard, such is utilized by the QR Factorization, which is A = QR or AP = QR. In order to execute it for matrix A, the following should be in place:

- A permutation matrix **P** : especially useful when not all the columns of matrix A is linearly independent.
- A matrix **Q** with orthonormal columns. To arrive at Q, there must be an orthogonal matrix U**,** produced from the Gramm Schmidt process wherein the norm of each column of U is calculated and then, each element of every column of U is divided by its respective norm.

- A square, invertible, upper triangular matrix **R**. To obtain R, multiply matrix A to the left of Q^T (transpose of Q).

A notable feature of QR Factorization is its orthonormal Q that works as a function that preserves length and angle. This provides for a more consistent environment on mathematical computations mainly in this day and age of computers where the need for Francis and Kublanovskaya's 1961 discovery of factorization is in demand.

Eigen Value/Vector Factorization

Another kind of basis is one that diagonalizes a matrix. Its benefits lie on simplified solutions and understandable methods. This concept leads to the EV Factorization represented as $A = V8V^{-1}$. To accomplish this task, the provisions below have to be at hand:

- A non-defective square matrix **A**. The number of eigen spaces must equal the number of columns in matrix A. This is ascertained after obtaining the eigen spaces.
- The **eigen values** symbolized by **λ** refer to the roots of a polynomial. To acquire these values, the determinant $(A - I\lambda) = 0$ must be performed employing the information on calculating determinants. These roots comprise the main diagonal of the diagonal matrix **λ** .
- The **eigen spaces** are the vectors that compose the matrix **V**. To achieve these eigen spaces, plug in the first eigen value λ on the matrix obtained from $A - I\lambda$. Reduce it to row echelon form which should reflect a matrix with all zeroes in the last row. Carry out the usual back substitution. The coordinates attach to a parameter make up an eigen vector. Repeat the same process for all the other eigen values.
- To get $\mathbf{V^{-1}}$, follow the rules of inverting matrices. If V is a square matrix, then it confirms that matrix A is not defective.

An advantage of EV Factorization is the diagonalization of a square matrix, which is key to finding solutions to some approaches on differential equations.

Orthogonal Factorization

A slight detour from EV Factorization is the Orthogonal Factorization, which corresponds to $A = Q\lambda Q^T$. In this case, if a linearly independent matrix A from the EV Factorization is a square, symmetric matrix, the matrix V can be substituted for the square orthonormal matrix Q. This practice eliminates the tedious undertaking of finding inverses since $Q^TQ = I$ results to $Q^T = Q^{-1}$. To implement this diagonalizing job, there should be:

- A matrix **A** described above.
- An orthogonal matrix **Q**.
- A matrix $\mathbf{Q^T}$ (transpose of Q).

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Singular Value Factorization

The all-encompassing factorization, which diagonalizes matrices without exception, including both the non-square matrices and the defective matrices, is the Singular Value Factorization (SVD): $A = UDV^{-1}$. Invented by Beltrami in 1873 and Jordan in 1874,its use propagated in 1967 when computers emerged. To pursue this activity for matrix A, it is necessary to have:

- Two orthogonal matrices **U** and **V**. The orthogonal matrices transform x and b in Ax = b into diagonalized systems, respectively.
- A diagonal matrix **D**, which has the same size as A. This should have nonnegative entries arranged from the highest to the lowest.
- The inverse of matrix V, $\mathbf{V^{-1}}$.

Besides, the flexibility of SVD Factorization, its use of orthogonal matrices, similar to a number of factorization types, makes it easy to solve systems of equations due to the consistency that orthogonal matrices provide. Additionally, it defines the norm of a matrix and the condition number of a matrix.

Conclusion

In retrospect, matrix factorization has become a fact of life in the modern period. The dynamic growth of technology and the swift progress of digitalization, impose the manipulation of matrices of various magnitudes. To rephrase the closing notes on page 489, the main issue on matrix factorization is knowing what type to employ to resolve a particular problem.

ADDENDUM:

Quick Guide/Reference Chart

Factorization Type	**Uses**	**Requirements**
LU Factorization $PA = LU$	**To solve** Square Matrices Underdetermined linear systems	**P** : Permutation matrix **A**: Given linear system **L**: Lower triangular matrix **U**: Matrix A in upper triangular row echelon form
QR Factorization $AP = QR$	**To obtain** Orthogonal basis **To approximate** Solutions to overdetermined linear systems	**A**: Given matrix **P**: Permutation matrix **Q**: Matrix with orthonormal columns **R**: An invertible, square, upper triangular matrix
EV Factorization $A = V\lambda V^{-1}$	**To transform** Square matrix into a diagonal system **To solve** Some differential equations	**A**: Given square matrix. λ: The root of a polynomial resulting from the determinant $(A-I\lambda) = 0$. **V**: Matrix made up of eigen vectors from the calculation of eigen spaces. $\mathbf{V^{-1}}$: The inverse of matrix V.
Orthogonal Factorization $A = Q\lambda Q^T$	**To transform** S Square, symmetric matrix into diagonal system using orthogonal matrix Q	**A**: Given square, symmetric matrix. **Q**: Orthogonal matrix. λ: Diagonal matrix with the same size as matrix A. $\mathbf{Q^{-1}}$: Inverse of matrix Q.
Singular Value Factorization $A = UDV^{-1}$	**To transform** S Matrices into diagonal system **Defines** Norm of the matrix Condition number of a matrix	**A**: Given matrix. **U** and **V**: Orthogonal matrices. **D**: Diagonal matrix with the same size as matrix A.